NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3131

ON THE KERNEL FUNCTION OF THE INTEGRAL EQUATION RELATING THE LIFT AND DOWNWASH DISTRIBUTIONS OF OSCILLATING FINITE WINGS IN SUBSONIC FLOW

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SUMMARY

This paper treats the kernel function of an integral equation that relates a known or prescribed downwash distribution to an unknown lift distribution for a harmonically oscillating finite wing in compressible subsonic flow. The kernel function is reduced to a form that can be accurately evaluated by separating the kernel function into two parts: a part in which the singularities are isolated and analytically expressed and a nonsingular part which may be tabulated. The form of the kernel function for the sonic case (Mach number of 1) is treated separately. In addition, results for the special cases of Mach number of 0 (incompressible case) and frequency of 0 (steady case) are given.

The derivation of the integral equation which involves this kernel function, originally performed elsewhere (see, for example, NACA Technical Memorandum 979) is reproduced as an appendix. A second appendix gives the reduction of the form of the kernel function obtained herein for the three-dimensional case to a known result of Possio for two-dimensional flow.

INTRODUCTION

The analytical determination of air forces on oscillating wings in subsonic flow has been a continuing problem for the past 30 years. Throughout the first and greater part of this time, efforts were directed mainly toward the determination of forces on wings in incompressible flow. These efforts have led to important closed-form solutions for rigid wings in two-dimensional flow (ref. 1), to solutions in terms of series of Legendre functions for distorting wings of circular plan form (refs. 2 and 3), and to many approximate, yet useful, results for wings of elliptic, rectangular, and triangular plan form (see, for example, refs. 4 to 12).

Although these results for incompressible flow play a highly significant roll in applications of unsteady aerodynamic theory, the advent of higher and higher speed aircraft during the last 15 years has brought a growing need for knowledge of the effect that the compressibility of air might have on unsteady air forces, or for analytically derived unsteady air forces based on a compressible medium. The transition to results for a compressible fluid from those for an incompressible fluid is not likely to be accomplished by applications of simple transformations or correction factors, such as the well-known Prandtl-Glauert factor for steady flow. This difficulty is associated with the fact that the time required for signals arising at one point in the medium to reach other points gives rise not only to changes in magnitudes of forces but also to additional phase lags between instantaneous positions, velocities, and accelerations of the wing and the corresponding instantaneous forces associated with these quantities. In order to obtain results for the compressible case, it therefore appears necessary to deal directly with the boundary-value problem for this case.

The boundary-value problem for a two-dimensional wing in compressible flow has been successfully attacked from two points of view. First, by consideration of an acceleration or pressure potential, Possio (ref. 13) reduced the problem to that of an integral equation relating a prescribed downwash distribution to an unknown lift distribution. The kernel of this integral equation, which is a rather abstruse function, was reduced to a form that, except at singular points, could be evaluated. Schwarz (ref. 14) later isolated and determined the analytic behavior of the singular points of Possio's results and made fairly extensive tables of the kernel function. These tabular values were used by various investigators (for examples, refs. 15 and 16) to obtain, by numerical procedures, initial tables of force and moment coefficients for oscillating wings in compressible subsonic flow.

The second successful approach to the solution of the boundary-value problem for a two-dimensional wing (see refs. 17 to 19) is achieved by a transformation to elliptic coordinates followed by a separation of variables that reduces the boundary-value problem from one in partial-differential equations to one in ordinary differential equations of the Mathieu type. The solutions turn out as infinite series in terms of Mathieu functions. Numerical results obtained recently by this procedure agree with results previously obtained by the numerical procedures using the kernel function (see, for example, ref. 20).

With regard to boundary-value problems for finite wings in compressible flow, it appears that the procedure of separation of variables could be a feasible approach only for wings of very special plan forms such as a circle or an ellipse. In any case the development of the appropriate mathematical functions for a particular plan form would become highly involved. On the other hand it appears that approximate procedures similar to those used for two-dimensional wings might afford an approach

to solutions of these problems which though laborious might be handled by routine numerical methods.

The kernel function of the integral equation relating pressure and downwash for the three-dimensional case appears as an improper integral that can be reduced and tabulated. Furthermore, its singularities can be isolated and determined so that use, similar to that made in the two-dimensional case, could be made of tabulated values of this function to obtain, by numerical procedures, aerodynamic forces for finite wings. The purpose of this paper is, therefore, to treat and discuss this kernel function.

SYMBOLS

c	velocity of sound
$H_0^{(2)},H_1^{(2)}$	Hankel functions of second kind of zero and first order, respectively
I_0,I_1	modified Bessel functions of first kind of zero and first order, respectively
J_0	Bessel function of first kind of zero order
K_0,K_1	modified Bessel functions of second kind of zero and first order, respectively
$K(x_o,y_o)$	kernel function of integral equation
$K'(x_o,y_o)$	singular part of $K(x_o,y_o)$
k	reduced-frequency parameter, $l\omega/V$
L_0,L_1	modified Struve function of zero and first order, respectively
$L(\xi,\eta)$	unknown lift distribution
l	reference length
M	Mach number, V/c
p	pressure

$r = \beta\sqrt{y_o^2 + z^2}$

S	region of xy-plane occupied by wing
t	time
V	forward velocity of wing
$\bar{w}(x,y)$	amplitude function of prescribed downwash, $w(x,y,t) = e^{i\omega t}\bar{w}(x,y)$
x,y,z,ξ,η	Cartesian coordinates

$x_o = x - \xi$

$y_o = y - \eta$

$\beta = \sqrt{1 - M^2}$

γ	Euler's constant

$\epsilon = \sqrt{x_o^2 + \beta^2 y_o^2}$

ϕ	velocity potential
ψ	acceleration potential
ρ	fluid density
ω	circular frequency of oscillation

$\bar{\omega} = \omega/V\beta^2$

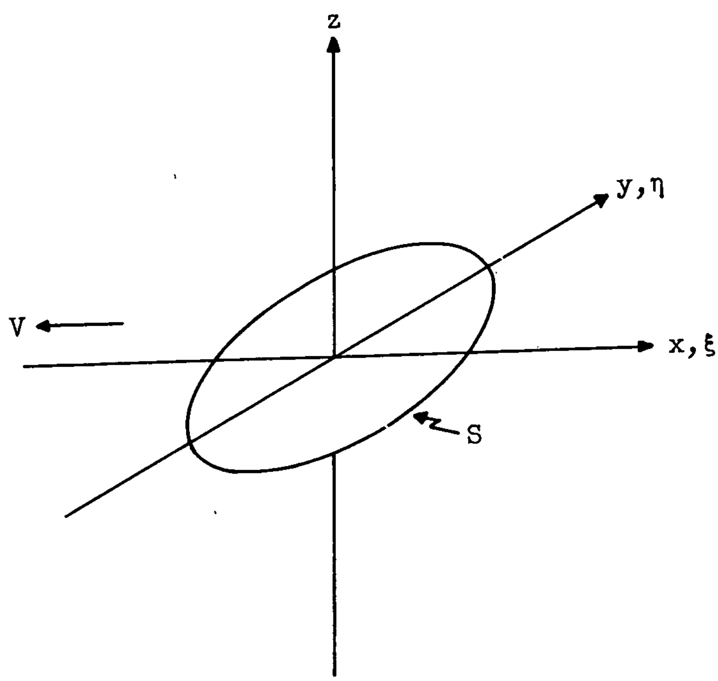
ANALYSIS

Integral Equation and Original Form of Kernel Function

The main purpose of this analysis is to treat the kernel function of an integral equation that relates a known or prescribed downwash distribution to an unknown lift distribution for a harmonically oscillating finite wing in compressible subsonic flow. The integral equation referred to can be obtained by employing the Prandtl acceleration potential to

treat linearized boundary-value problems for oscillating finite wings by means of doublet distributions. Derivation of this integral equation from the linearized boundary-value problem for a wing is a preliminary task that has been done elsewhere (see, for example, ref. 21), but is reproduced herein as an appendix for the sake of completeness.

In keeping with the concepts of linear theory, the wing is considered a plane impenetrable surface S which lies nearly in the xy-plane as indicated in sketch 1:



Sketch 1

The x,y,z coordinate system and the surface S are assumed to move in the negative x-direction at a uniform velocity V.

In terms of these coordinates the integral equation may be formally written as

$$\overline{w}(x,y) = \frac{1}{4\pi} \iint_S L(\xi,\eta)\, K(x_o,y_o)\, d\xi\, d\eta \tag{1}$$

where $\bar{w}(x,y)$ is the amplitude function of the prescribed downwash, $K(x_o,y_o) = K(x - \xi, y - \eta)$ is the kernel function and physically represents the contribution to downwash at a field point (x,y) due to a pulsating pressure doublet of unit strength located at any point (ξ,η), and $L(\xi,\eta)$ is the unknown lift distribution or local doublet strength.

The kernel function may be mathematically defined by the following improper integral expression (see eq. (A12), appendix A):

$$K(x_o,y_o) = \lim_{z \to 0} \frac{\partial^2}{\partial z^2} e^{-\frac{i\omega x_o}{V}} \int_{-\infty}^{x_o} \frac{e^{i\bar{\omega}\left(\lambda - M\sqrt{\lambda^2+\beta^2 y_o{}^2+\beta^2 z^2}\right)}}{\sqrt{\lambda^2 + \beta^2 y_o{}^2 + \beta^2 z^2}} \, d\lambda \tag{2}$$

where M is Mach number, $\beta = \sqrt{1 - M^2}$, $\bar{\omega} = \omega/V\beta^2$, ω is the circular frequency of oscillation, V is the velocity, and λ is the variable of integration. Evaluation of this integral constitutes a main difficulty in obtaining aerodynamic coefficients for oscillating finite wings in compressible flow. The present analysis is therefore devoted to reducing it to a form that can be accurately evaluated by numerical procedures combined with the use of tables of certain tabulated functions. The form and order of all its singularities are determined and an expression for the kernel function is derived in which the singularities are isolated.

Reduction of the Kernel Function

In considering the reduction of the kernel function $K(x_o,y_o)$, the integral involved can, for convenience, be written as the sum of two integrals, namely

$$\int_{-\infty}^{x_o} \frac{e^{i\bar{\omega}\left(\lambda - M\sqrt{\lambda^2+r^2}\right)}}{\sqrt{\lambda^2 + r^2}} \, d\lambda = \int_0^{\infty} \frac{e^{-i\bar{\omega}\left(\lambda + M\sqrt{\lambda^2+r^2}\right)}}{\sqrt{\lambda^2 + r^2}} \, d\lambda + \int_0^{x_o} \frac{e^{i\bar{\omega}\left(\lambda - M\sqrt{\lambda^2+r^2}\right)}}{\sqrt{\lambda^2 + r^2}} \, d\lambda \tag{3}$$

Therefore,

$$K(x_o,y_o) = \lim_{z \to 0} \frac{\partial^2}{\partial z^2} e^{-\frac{i\omega x_o}{V}} (F) = \lim_{z \to 0} \frac{\partial^2}{\partial z^2} e^{-\frac{i\omega x_o}{V}} \left(F_1 + F_2\right) \tag{4}$$

where

$$F_1 = \int_0^{\infty} \frac{e^{-i\bar{\omega}\left(\lambda+M\sqrt{\lambda^2+r^2}\right)}}{\sqrt{\lambda^2 + r^2}}\,d\lambda \tag{5}$$

and

$$F_2 = \int_0^{x_o} \frac{e^{i\bar{\omega}\left(\lambda-M\sqrt{\lambda^2+r^2}\right)}}{\sqrt{\lambda^2 + r^2}}\,d\lambda \tag{6}$$

and where $r = \beta\sqrt{y_o^2 + z^2}$.

The integrals F_1 and F_2 are treated separately in succeeding sections. The final forms are given in equations (15) and (19), respectively.

Evaluation of F_1.- The integral F_1 can be converted to a form that can be more easily handled by writing

$$F_1 = \int_0^{\infty} e^{-i\bar{\omega}\lambda}\,\frac{e^{-i\bar{\omega}M\sqrt{\lambda^2+r^2}}}{\sqrt{\lambda^2 + r^2}}\,d\lambda$$

and introducing the following relation (see p. 416 of ref. 22)

$$\begin{aligned}\frac{e^{-i\bar{\omega}M\sqrt{\lambda^2+r^2}}}{\sqrt{\lambda^2 + r^2}} &= \int_0^{\infty} J_0(T\lambda)\,\frac{e^{-r\sqrt{T^2-M^2\bar{\omega}^2}}}{\sqrt{T^2 - M^2\bar{\omega}^2}}\,T\,dT \\ &= \int_{M\bar{\omega}}^{\infty} J_0(T\lambda)\,\frac{e^{-r\sqrt{T^2-M^2\bar{\omega}^2}}}{\sqrt{T^2 - M^2\bar{\omega}^2}}\,T\,dT - i\int_0^{M\bar{\omega}} J_0(T\lambda)\,\frac{e^{-ir\sqrt{M^2\bar{\omega}^2-T^2}}}{\sqrt{M^2\bar{\omega}^2 - T^2}}\,T\,dT\end{aligned} \tag{7}$$

In the first integral of these last two integrals make the subsitution

$$\sqrt{T^2 - M^2\bar{\omega}^2} = \tau$$

and in the second integral make the substitution

$$\sqrt{M^2\bar{\omega}^2 - T^2} = \tau$$

Then

$$\frac{e^{-i\bar{\omega}M\sqrt{\lambda^2+r^2}}}{\sqrt{\lambda^2 + r^2}} = \int_0^\infty e^{-r\tau}\ J_0\left(\lambda\sqrt{\tau^2 + M^2\bar{\omega}^2}\right)d\tau\ -$$

$$i\int_0^{M\bar{\omega}} e^{-ir\tau}\ J_0\left(\lambda\sqrt{M^2\bar{\omega}^2 - \tau^2}\right)d\tau \tag{8}$$

(It is of interest to note, in the expression on the left of eq. (8), that λ and r appear in the same manner. The roles of these two quantities could therefore be interchanged in the expression on the right.)

With use of equation (8), the equation for F_1 can be written as

$$F_1 = \int_0^\infty e^{-i\bar{\omega}\lambda}\ d\lambda\left[\int_0^\infty e^{-r\tau}\ J_0\left(\lambda\sqrt{\tau^2 + M^2\bar{\omega}^2}\right)d\tau\ -\right.$$

$$\left. i\int_0^{M\bar{\omega}} e^{-ir\tau}\ J_0\left(\lambda\sqrt{M^2\bar{\omega}^2 - \tau^2}\right)d\tau\right] \tag{9}$$

Changing the order of integration in each integral (which is a legitimate step because the integrands involved satisfy the continuity conditions required for such operations) leads to the following expression for F_1:

$$F_1 = \int_0^\infty e^{-r\tau}\ d\tau\left[\int_0^\infty e^{-i\bar{\omega}\lambda}\ J_0\left(\lambda\sqrt{\tau^2 + M^2\bar{\omega}^2}\right)d\lambda\right] -$$

$$i\int_0^{M\bar{\omega}} e^{-ir\tau}\ d\tau\left[\int_0^\infty e^{-i\bar{\omega}\lambda}\ J_0\left(\lambda\sqrt{M^2\bar{\omega}^2 - \tau^2}\right)d\lambda\right] \tag{10}$$

The integrals within the brackets in equation (10) may be evaluated from tables of Fourier or Laplace transforms as (see, for example, pair no. 55 of appendix III of ref. 23)

$$\int_0^\infty e^{-i\bar{\omega}\lambda}\, J_0\left(\lambda\sqrt{\tau^2 + M^2\bar{\omega}^2}\right)d\lambda = \frac{1}{\sqrt{\tau^2 - \beta^2\bar{\omega}^2}}$$

$$\int_0^\infty e^{-i\bar{\omega}\lambda}\, J_0\left(\lambda\sqrt{M^2\bar{\omega}^2 - \tau^2}\right)d\lambda = \frac{-i}{\sqrt{\tau^2 + \beta^2\bar{\omega}^2}}$$

so that

$$F_1 = \int_0^\infty \frac{e^{-r\tau}}{\sqrt{\tau^2 - \beta^2\bar{\omega}^2}}\, d\tau - \int_0^{M\bar{\omega}} \frac{e^{-ir\tau}}{\sqrt{\tau^2 + \beta^2\bar{\omega}^2}}\, d\tau \tag{11}$$

The first integral in equation (11) can be written as

$$\int_0^\infty \frac{e^{-r\tau}}{\sqrt{\tau^2 - \beta^2\bar{\omega}^2}}\, d\tau = \int_{\beta\bar{\omega}}^\infty \frac{e^{-r\tau}}{\sqrt{\tau^2 - \beta^2\bar{\omega}^2}}\, d\tau - i\int_0^{\beta\bar{\omega}} \frac{e^{-r\tau}}{\sqrt{\beta^2\bar{\omega}^2 - \tau^2}}\, d\tau$$

or

$$\int_0^\infty \frac{e^{-r\tau}}{\sqrt{\tau^2 - \beta^2\bar{\omega}^2}}\, d\tau = \int_0^\infty e^{-\beta\bar{\omega}r\cosh\theta}\, d\theta - \frac{i}{2}\int_{-\pi/2}^{\pi/2} e^{-\beta\bar{\omega}r\cos\theta}\, d\theta \tag{11a}$$

The first integral on the right of equation (11a) is given on page 181 of reference 22 as

$$\int_0^\infty e^{-\beta\bar{\omega}r\cosh\theta}\, d\theta = K_0(\beta\bar{\omega}r)$$

where K_0 is the modified Bessel function of the second kind of zero order. The second integral on the right of equation (11a) is given on page 338 of reference 22 as

$$-\frac{i}{2}\int_{-\pi/2}^{\pi/2} e^{-\beta\bar{\omega}r\cos\theta}\, d\theta = -\frac{\pi i}{2}\left[I_0(\beta\bar{\omega}r) - L_0(\beta\bar{\omega}r)\right]$$

where I_0 is the modified Bessel function of the first kind of zero order and L_0 is the modified Struve function of zero order. Then, the first integral of equation (11) can be written as

$$\int_0^\infty \frac{e^{-r\tau}}{\sqrt{\tau^2 - \beta^2\bar{\omega}^2}}\,d\tau = K_0(\beta\bar{\omega}r) - \frac{\pi i}{2}\left[I_0(\beta\bar{\omega}r) - L_0(\beta\bar{\omega}r)\right]$$

$$= K_0\left(\frac{\omega}{V}\sqrt{y_o^2 + z^2}\right) - \frac{\pi i}{2}\left[I_0\left(\frac{\omega}{V}\sqrt{y_o^2 + z^2}\right) - L_0\left(\frac{\omega}{V}\sqrt{y_o^2 + z^2}\right)\right] \tag{12}$$

Note that the end result indicated in equation (12) is independent of Mach number. The second integral in equation (11) may be written in another form as

$$\int_0^{M\bar{\omega}} \frac{e^{-ir\tau}}{\sqrt{\tau^2 + \beta^2\bar{\omega}^2}}\,d\tau = \int_0^{M/\beta} \frac{e^{-i\left(\frac{\omega}{V}\sqrt{y_o^2+z^2}\right)\tau}}{\sqrt{1 + \tau^2}}\,d\tau \tag{13}$$

This integral has not been reduced to closed form; however, it is nonsingular and can be readily handled by numerical methods.

Combining equations (12) and (13) gives the following expression for F_1:

$$F_1 = K_0\left(\frac{\omega}{V}\sqrt{y_o^2 + z^2}\right) - \frac{\pi i}{2}\left[I_0\left(\frac{\omega}{V}\sqrt{y_o^2 + z^2}\right) - L_0\left(\frac{\omega}{V}\sqrt{y_o^2 + z^2}\right)\right] - \int_0^{M/\beta} \frac{e^{-i\left(\frac{\omega}{V}\sqrt{y_o^2+z^2}\right)\tau}}{\sqrt{1 + \tau^2}}\,d\tau \tag{14}$$

By performing the differentiations indicated in equation (4), there is obtained for the first part of equation (4) the following expression:

$$\lim_{z\to 0} \frac{\partial^2 F_1}{\partial z^2} = \frac{\omega}{V|y_o|}\left\{-K_1\left(\frac{\omega}{V}|y_o|\right) - \frac{\pi i}{2}\left[I_1\left(\frac{\omega}{V}|y_o|\right) - L_1\left(\frac{\omega}{V}|y_o|\right)\right] + \right.$$

$$\left. \frac{i}{\beta}\, e^{-\frac{iM\omega|y_o|}{\beta V}} - \frac{\omega}{V}|y_o| \int_0^{M/\beta} \sqrt{1+\tau^2}\, e^{-i\frac{\omega}{V}|y_o|\tau}\, d\tau \right\} \tag{15}$$

All terms of this expression other than the integral may be evaluated at small intervals of y_o from existing tables, except at $y_o = 0$ where the function is singular. The integral is well-behaved and can be accurately evaluated by numerical or approximate procedures. The type and order of the singularities at $y_o = 0$ are discussed in a later section.

Evaluation of F_2.- In order to reduce the integral F_2, equation (6), it is convenient to make the substitution

$$\lambda = r \sinh \theta \tag{16}$$

so that

$$F_2 = \int_0^{\sinh^{-1}\frac{x_o}{r}} e^{i\bar{\omega}r(\sinh\theta - M\cosh\theta)}\, d\theta \tag{17}$$

Noting that z appears only in r and performing the differentiations indicated in equation (4) yields

$$\left(\frac{\partial^2 F_2}{\partial z^2}\right)_{z=0} = \frac{i\bar{\omega}\beta}{|y_o|}\int_0^{\sinh^{-1}\frac{x_o}{\beta|y_o|}} (\sinh\theta - M\cosh\theta) e^{i\bar{\omega}\beta|y_o|(\sinh\theta - M\cosh\theta)}\, d\theta - \frac{x_o}{y_o^2\sqrt{x_o^2+\beta^2 y_o^2}}\, e^{i\bar{\omega}\left(x_o - M\sqrt{x_o^2+\beta^2 y_o^2}\right)}$$

$$= -\beta^2\left\{\frac{x_o}{\beta^2 y_o^2\sqrt{x_o^2+\beta^2 y_o^2}}\, e^{i\bar{\omega}\left(x_o - M\sqrt{x_o^2+\beta^2 y_o^2}\right)} - \frac{i\bar{\omega}}{M\beta|y_o|}\int_0^{\sinh^{-1}\frac{x_o}{\beta|y_o|}} \left[\beta^2\cosh\theta - (\cosh\theta - M\sinh\theta)\right] e^{i\bar{\omega}\beta|y_o|(\sinh\theta - M\cosh\theta)}\, d\theta\right\}$$

$$= -\beta^2\left\{\frac{x_o\, e^{i\bar{\omega}\left(x_o - M\sqrt{x_o^2+\beta^2 y_o^2}\right)}}{\beta^2 y_o^2\sqrt{x_o^2+\beta^2 y_o^2}} + \frac{1}{M\beta^2 y_o^2}\left[e^{i\bar{\omega}\left(x_o - M\sqrt{x_o^2-\beta^2 y_o^2}\right)} - e^{-i\bar{\omega}M\beta|y_o|}\right] - \frac{i\bar{\omega}\beta}{M|y_o|}\int_0^{\sinh^{-1}\frac{x_o}{\beta|y_o|}} \cosh\theta\, e^{i\bar{\omega}\beta|y_o|(\sinh\theta - M\cosh\theta)}\, d\theta\right\}$$

$$\tag{18}$$

or, by reverting completely to Cartesian coordinates through equation (16), there is obtained

$$\left(\frac{\partial^2 F_2}{\partial z^2}\right)_{z=0} = -\left\{\frac{x_o\, e^{i\bar{\omega}\left(x_o - M\sqrt{x_o^2+\beta^2 y_o^2}\right)}}{y_o^2\sqrt{x_o^2+\beta^2 y_o^2}} + \frac{1}{My_o^2}\left[e^{i\bar{\omega}\left(x_o - M\sqrt{x_o^2+\beta^2 y_o^2}\right)} - e^{-i\bar{\omega}M\beta|y_o|}\right] - \frac{i\beta^2\bar{\omega}}{My_o^2}\int_0^{x_o} e^{i\bar{\omega}\left(\lambda - M\sqrt{\lambda^2+\beta^2 y_o^2}\right)}d\lambda\right\} \tag{19}$$

This expression vanishes, as it should, for $x_o = 0$ and like that in equation (15) has singularities at $y_o = 0$ which, also, will be handled in a later section. The integral that remains, like the integral remaining in equation (15), is nonsingular and simple in form and can be readily evaluated by numerical procedures.

<u>Expression for the kernel in terms of nondimensional length variables.</u>- Equations (15) and (19) can now be combined to give a reduced form of the kernel function $K(x_o,y_o)$. However, in application, the variables x_o and y_o are employed, for convenience, in nondimensional form. This is accomplished by considering these variables in a new sense to mean that they have been referred to some chosen length l and by introducing the reduced-frequency parameter $k = l\omega/V$. The variables will be used in this new sense throughout the remainder of the paper. The kernel can be written in terms of these nondimensional variables as

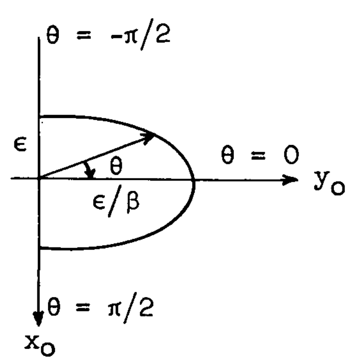
$$\begin{aligned}K(x_o,y_o) &= e^{-ikx_o}\frac{\partial^2}{\partial z^2}(F_1+F_2)_{z=0}\\ &= \frac{k^2}{l^2}e^{-ikx_o}\left\{-\frac{1}{k|y_o|}K_1(k|y_o|) - \frac{\pi i}{2k|y_o|}\left[I_1(k|y_o|) - L_1(k|y_o|)\right] + \frac{iMk|y_o| + \beta}{M\beta(ky_o)^2}e^{-\frac{iMk|y_o|}{\beta}} - \right.\\ &\quad \int_0^{M/\beta}\sqrt{1+\tau^2}\,e^{-ik|y_o|\tau}\,d\tau - \frac{Mkx_o + \sqrt{(kx_o)^2+\beta^2(ky_o)^2}}{M(ky_o)^2\sqrt{(kx_o)^2+\beta^2(ky_o)^2}}e^{\frac{i}{\beta^2}\left[kx_o - M\sqrt{(kx_o)^2+\beta^2(ky_o)^2}\right]} + \\ &\quad \left.\frac{i}{M(ky_o)^2}\int_0^{kx_o} e^{\frac{i}{\beta^2}\left[\lambda - M\sqrt{\lambda^2+\beta^2(ky_o)^2}\right]}d\lambda\right\}\end{aligned} \tag{20}$$

Note that this expression for $K(x_o,y_o)$ can be considered as a function of only three parameters, namely, $k|y_o|$, kx_o, and M. To be more specific, the first two terms are functions only of $k|y_o|$; the next two terms are functions of $k|y_o|$ and M; and the last two terms are functions of $k|y_o|$, kx_o, and M.

Equation (20) constitutes the principal result of this paper. Some partial checks as to its correctness are: (1) For $k = 0$, it reduces, as discussed subsequently, to the downwash of a pressure doublet in steady flow and (2) an integration with regard to the y-direction between the limits $-\infty$ to $+\infty$ yields Possio's result for the two-dimensional case. This integration is carried out in appendix B. Other special forms of the kernel function for $M = 1$, $M = 0$, and $k = 0$ are derived in subsequent sections. In the section immediately following, the orders and types of the singularities of the kernel function are discussed.

Discussion of the Singularities of the Kernel Function

As previously indicated, the kernel function becomes singular or indeterminate at $y_o = 0$. The forms that the kernel function takes when it becomes singular are of particular importance in applications to lifting surface theory. It is therefore desirable to extract and treat the singularities separately.



Sketch 2

This extraction can be conveniently made by considering the value of $K(x_o,y_o)$, equation (20), at points on the semicircumference of a small ellipse (see sketch), the polar equation of which may be written as

$$\left.\begin{aligned} x_o &= \epsilon \sin \theta \\ y_o &= \frac{\epsilon}{\beta} \cos \theta \end{aligned}\right\} \qquad (21)$$

where, because of the symmetry of $K(x_o,y_o)$ with respect to y_o, only the limits $-\pi/2 \leqq \theta \leqq \pi/2$ need be examined. Note that in these equations values of θ in the range $-\pi/2 \leqq \theta < 0$ correspond to field points ahead of or upstream from the doublet position and values of θ

in the range $0 < \theta \leqq \pi/2$, to field points behind or downstream from the doublet position. In particular, $\theta = \pi/2$ corresponds to points directly behind or in the wake of the doublet.

After substituting these expressions for x_o and y_o into equation (20), the results may be written as

$$K(\epsilon,\theta) = \frac{\beta^2 e^{-ik\epsilon \sin\theta}}{l^2\epsilon^2\cos^2\theta}\left\{-\frac{k\epsilon\cos\theta}{\beta}K_1\left(\frac{k\epsilon\cos\theta}{\beta}\right) - \right.$$

$$\frac{i\pi k\epsilon\cos\theta}{2\beta}\left[I_1\left(\frac{k\epsilon\cos\theta}{\beta}\right) - L_1\left(\frac{k\epsilon\cos\theta}{\beta}\right)\right] + \frac{e^{-\frac{ikM\epsilon\cos\theta}{\beta^2}}}{M} -$$

$$\frac{e^{\frac{ik\epsilon(\sin\theta - M)}{\beta^2}}}{M} + \frac{ik\epsilon\cos\theta}{\beta^2}e^{-\frac{ikM\epsilon\cos\theta}{\beta^2}} -$$

$$\sin\theta\, e^{\frac{ik\epsilon(\sin\theta - M)}{\beta^2}} - \frac{k^2\epsilon^2\cos^2\theta}{\beta^2}\int_0^{M/\beta}\sqrt{1+\tau^2}\, e^{-\left(\frac{ik\epsilon\cos\theta}{\beta}\right)\tau}\, d\tau +$$

$$\left.\frac{ik}{M}\int_0^{\epsilon\sin\theta} e^{\frac{ik}{\beta^2}\left(\lambda - M\sqrt{\lambda^2+\epsilon^2\cos^2\theta}\right)}\, d\lambda\right\} \tag{22}$$

With the use of the following series expressions for $K_1(z)$ and $\left[I_1(z) - L_1(z)\right]$ (which can be obtained from ref. 22 - for K_1, see p. 80; for I_1, see p. 77; and for L_1, see p. 329):

$$K_1(z) = \left(\gamma + \log\frac{z}{2}\right)\left(\frac{z}{2} + \frac{z^3}{16} + \frac{z^5}{384} + \ldots\right) +$$

$$\frac{1}{z} - \left(\frac{z}{4} + \frac{5z^3}{64} + \frac{5z^5}{1152} + \ldots\right) \tag{23}$$

where γ is Euler's constant ($\gamma = 0.5772157$), and

$$\left[I_1(z) - L_1(z)\right] = \frac{z}{2} - \frac{2z^2}{3\pi} + \frac{z^3}{16} - \frac{2z^4}{45\pi} + \frac{z^5}{384} + \ldots \tag{24}$$

it is found that for vanishingly small values of ϵ the limiting value of the expression for $K(\epsilon,\theta)$ in equation (22) is for $M < 1$

$$K(\epsilon,\theta) \approx \frac{e^{-ik\epsilon \sin\theta}}{l^2}\left\{\frac{-\beta^2}{\epsilon^2(1 - \sin\theta)} + \frac{ik}{\epsilon} - \frac{k^2}{2}\log\frac{k\epsilon(1 - \sin\theta)}{2(1 - M)} - \frac{k^2}{2}\left[\gamma - \frac{1}{2} - \frac{1}{\beta^2}\left(M - \sin\theta - \frac{i\pi\beta^2}{2}\right)\right] + O(\epsilon^n)\right\} \tag{25}$$

where $O(\epsilon^n)$ represents terms of order ϵ^n for $n \geqq 1$. Expressed in terms of x_o and y_o, equation (25) becomes

$$K(x_o,y_o) \approx \frac{e^{-ikx_o}}{l^2}\left\{\frac{-\left(x_o + \sqrt{x_o^2 + \beta^2 y_o^2}\right)}{y_o^2\sqrt{x_o^2 + \beta^2 y_o^2}} + \frac{ik}{\sqrt{x_o^2 + \beta^2 y_o^2}} - \frac{k^2}{2}\log\frac{k\left(\sqrt{x_o^2 + \beta^2 y_o^2} - x_o\right)}{2(1 - M)} - \frac{k^2}{2}\left[\gamma - \frac{1}{2} - \frac{1}{\beta^2}\left(M - \frac{x_o}{\sqrt{x_o^2 + \beta^2 y_o^2}} - \frac{i\pi\beta^2}{2}\right)\right] + O(\epsilon^n)\right\} \tag{26}$$

Examination of equation (25) shows that the kernel function $K(\epsilon,\theta)$ has singularities with respect to $\epsilon = \sqrt{x_o^2 + \beta^2 y_o^2}$ as follows:

$$-\frac{f_1(\theta)}{\epsilon^2} \quad ; \quad \frac{ik}{\epsilon} \quad ; \quad -\frac{k^2}{2}\left[f_2(\theta) + \log\epsilon\right] \tag{27}$$

where, from equation (25),

$$
\left.
\begin{aligned}
f_1(\theta) &= \frac{\beta^2}{1 - \sin\theta} = \frac{\beta^2(1 + \sin\theta)}{\cos^2\theta} \\
f_2(\theta) &= \log\frac{k(1 - \sin\theta)}{2(1 - M)} = \log\frac{k\cos^2\theta}{2(1 - M)(1 + \sin\theta)}
\end{aligned}
\right\}
\tag{28}
$$

Although being of no particular significance in applications, it is of interest to note that the quantities f_1 and f_2 each have minimum values $\left(|f_1|_{min} = \frac{\beta^2}{2} \text{ and } |f_2|_{min} = \log\frac{k}{1 - M}\right)$ at $\theta = -\pi/2$, which corresponds to points directly ahead of the doublet position; and, as θ increases from $-\pi/2$ to $+\pi/2$, the values of these quantities continuously increase from these minimum values to infinite quantities as follows:

$$
\left.
\begin{aligned}
\left|f_1\left(\frac{\pi}{2}\right)\right| &= \lim_{\delta\to 0}\left|\frac{\beta^2\left[1 + \sin\left(\frac{\pi}{2} - \delta\right)\right]}{\cos^2\left(\frac{\pi}{2} - \delta\right)}\right| = \lim_{\delta\to 0}\left|\frac{2\beta^2}{\delta^2}\right| \\
\left|f_2\left(\frac{\pi}{2}\right)\right| &= \lim_{\delta\to 0}\left|\log\frac{k\cos^2\left(\frac{\pi}{2} - \delta\right)}{2(1 - M)\left[1 + \sin\left(\frac{\pi}{2} - \delta\right)\right]}\right| = \lim_{\delta\to 0}\left|\log\frac{k\delta^2}{4(1 - M)}\right|
\end{aligned}
\right\}
\tag{29}
$$

Thus $K(x_o,y_o)$ is singular for $\theta = \pi/2$ even when the distance ϵ from the doublet is not necessarily of zero order. This implies that the doublet produces a wake of discontinuous downwash that extends downstream from the doublet position to infinity.

With knowledge of the singularities involved in the kernel function $K(x_o,y_o)$, an expression can be written in which the kernel is separated into a singular part and a nonsingular part (as was done by Schwarz, ref. 14, for the two-dimensional case) as follows

$$
K(x_o,y_o) \equiv \left[K(x_o,y_o) - K'(x_o,y_o)\right] + K'(x_o,y_o)
\tag{30}
$$

where $K(x_o,y_o)$ is defined in equation (20) or (22) and

$$K'(x_o,y_o) = \frac{e^{-ikx_o}}{l^2}\left[-\frac{\sqrt{x_o^2 + \beta^2 y_o^2} + x_o}{y_o^2\sqrt{x_o^2 + \beta^2 y_o^2}} + \frac{ik}{\sqrt{x_o^2 + \beta^2 y_o^2}} - \frac{k^2}{2}\log\frac{k\left(\sqrt{x_o^2 + \beta^2 y_o^2} - x_o\right)}{2(1 - M)}\right] \tag{31}$$

or in terms of ϵ and θ, introduced by equations (21),

$$K'(\epsilon,\theta) = \frac{e^{-ik\epsilon \sin\theta}}{l^2}\left[-\frac{\beta^2}{\epsilon^2(1 - \sin\theta)} + \frac{ik}{\epsilon} - \frac{k^2}{2}\log\frac{k\epsilon(1 - \sin\theta)}{2(1 - M)}\right] \tag{32}$$

The term $\left[K(x_o,y_o) - K'(x_o,y_o)\right]$ in equation (30) is a continuous function for all values of k, x_o, and y_o and for values of M in the range $0 \leqq M \leqq 1$. The term $K'(x_o,y_o)$ is discontinuous at the doublet position $(x_o = 0,\ y_o = 0)$ and at all points in the wake $(x_o > 0,\ y_o = 0)$. It is to be noted, however, that each term of $K'(x_o,y_o)$ is integrable with respect to y_o or with respect to $\eta = y - y_o$, a fact that may be useful in some numerical methods.

Treatment of the Sonic Case

Because of its special nature, the borderline case, $M = 1$, between subsonic and supersonic flow deserves and requires separate treatment.

As $M \longrightarrow 1$, the expression for the kernel function given in equation (20) becomes indeterminate. It is possible, however, to obtain conditional limiting values for the kernel by considering the integral F, equation (4), and breaking it into two integrals, F_1 and F_2, as was done for the general case.

With regard to F_1, its limiting value and the value of its derivatives with respect to z at $z = 0$ can be shown to be zero as $M \longrightarrow 1$. From the form of F_1 given by equation (14),

$$\lim_{M\to 1} F_1 = \lim_{M\to 1}\left\{K_0\left(\frac{\omega}{V}\sqrt{y_o{}^2 + z^2}\right) - \frac{\pi i}{2}\left[I_0\left(\frac{\omega}{V}\sqrt{y_o{}^2 + z^2}\right) - L_0\left(\frac{\omega}{V}\sqrt{y_o{}^2 + z^2}\right)\right] - \int_0^{M/\beta} \frac{e^{-i\left(\frac{\omega}{V}\sqrt{y_o{}^2+z^2}\right)\tau}}{\sqrt{1 + \tau^2}}\,d\tau\right\}$$

$$= K_0\left(\frac{\omega}{V}\sqrt{y_o{}^2 + z^2}\right) - \frac{\pi i}{2}\left[I_0\left(\frac{\omega}{V}\sqrt{y_o{}^2 + z^2}\right) - L_0\left(\frac{\omega}{V}\sqrt{y_o{}^2 + z^2}\right)\right] - \int_0^{\infty} \frac{\cos\left[\left(\frac{\omega}{V}\sqrt{y_o{}^2 + z^2}\right)\tau\right]}{\sqrt{1 + \tau^2}}\,d\tau + i\int_0^{\infty} \frac{\sin\left[\left(\frac{\omega}{V}\sqrt{y_o{}^2 + z^2}\right)\tau\right]}{\sqrt{1 + \tau^2}}\,d\tau \tag{33}$$

But since (see ref. 22, p. 172)

$$-\int_0^{\infty} \frac{\cos \zeta\tau}{\sqrt{1 + \tau^2}}\,d\tau = -K_0(\zeta) \tag{34}$$

and (see ref. 22, p. 332)

$$i\int_0^{\infty} \frac{\sin \zeta\tau}{\sqrt{1 + \tau^2}}\,d\tau = \frac{\pi i}{2}\left[I_0(\zeta) - L_0(\zeta)\right] \tag{35}$$

it may be concluded from equation (33) that

$$\lim_{M\to 1} F_1 = \lim_{M\to 1}\left(\frac{\partial^2 F_1}{\partial z^2}\right) = 0 \tag{36}$$

The total contribution to $K(x_o,y_o)$ at $M = 1$, therefore, arises from the limit of F_2, equation (6), as $M \longrightarrow 1$. The limiting form of F_2 may be written in terms of nondimensional coordinates as

$$\lim_{M\to 1} F_2 = \lim_{M\to 1} \int_0^{x_o} \frac{e^{\frac{ik}{\beta^2}\left[\lambda - M\sqrt{\lambda^2+\beta^2\left(y_o^2+z^2\right)}\right]}}{\sqrt{\lambda^2 + \beta^2\left(y_o^2 + z^2\right)}} d\lambda \tag{37}$$

In approaching the limit $M = 1$ (from the subsonic side) in equation (37), it is convenient to replace M by

$$M = 1 - \epsilon$$

where ϵ is infinitesimally small so that

$$\beta^2 = (1 - M)(1 + M) = \epsilon(2 - \epsilon) \approx 2\epsilon$$

With this approximation, equation (37) may be written as

$$\lim_{M\to 1} F_2 = \lim_{\epsilon\to 0} \int_0^{x_o} \frac{e^{\frac{ik}{2\epsilon}\left\{\lambda - |\lambda|(1-\epsilon)\left[1+\frac{\epsilon\left(y_o^2+z^2\right)}{\lambda^2}\right]\right\}}}{\sqrt{\lambda^2 + 2\epsilon\left(y_o^2 + z^2\right)}} d\lambda$$

$$= \int_0^{x_o} \frac{e^{\frac{ik}{2}\left(\lambda - \frac{y_o^2+z^2}{\lambda}\right)}}{\lambda} d\lambda \qquad (\text{for } x_o > 0) \tag{38}$$

From physical considerations, the right side of equation (38) is to be considered zero for $x_o \leqq 0$. This is in keeping with results that would be obtained if the limit under consideration were sought from theory of supersonic flow, $M > 1$.

The integral in equation (38) cannot be completely expressed in terms of known functions. Furthermore, since it is singular at its lower limit, further treatment is required to reduce it to a form such that its derivatives with respect to z can be numerically evaluated. For this purpose the integral may be written as two integrals, namely

$$\left(F_2\right)_{M=1} = F_2' + F_2'' \tag{39}$$

where

$$F_2' = \int_0^{\sqrt{y_o^2+z^2}} \frac{e^{\frac{ik}{2}\left(\lambda - \frac{y_o^2+z^2}{\lambda}\right)}}{\lambda} d\lambda \tag{40}$$

and

$$F_2'' = \int_{\sqrt{y_o^2+z^2}}^{x_o} \frac{e^{\frac{ik}{2}\left(\lambda - \frac{y_o^2+z^2}{\lambda}\right)}}{\lambda} d\lambda \tag{41}$$

The limits of integration in equation (40) are so chosen that the integral in this equation can be reduced to a known form by making the substitution

$$\lambda = \sqrt{\tau^2 + \left(y_o^2 + z^2\right)} - \tau \quad \text{or} \quad \tau = \frac{1}{2}\left(\frac{y_o^2 + z^2}{\lambda} - \lambda\right)$$

Thus,

$$F_2' = \int_0^{\infty} \frac{e^{-ik\tau}}{\sqrt{\tau^2 + \left(y_o^2 + z^2\right)}} d\tau = \int_0^{\infty} \frac{e^{-i\left(k\sqrt{y_o^2+z^2}\right)\tau}}{\sqrt{1 + \tau^2}} d\tau \tag{42}$$

Equation (42) may be written in terms of the integrals involved in F_1 (see eqs. (34) and (35)), namely,

$$F_2' = K_0\left(k\sqrt{y_o^2 + z^2}\right) - \frac{\pi i}{2}\left[I_0\left(k\sqrt{y_o^2 + z^2}\right) - L_0\left(k\sqrt{y_o^2 + z^2}\right)\right] \tag{43}$$

Differentiating this result twice with respect to z and then setting $z = 0$ gives

$$\left(\frac{\partial^2 F_2'}{\partial z^2}\right)_{z=0} = \frac{k^2}{l^2}\left\{-\frac{1}{k|y_o|}K_1\left(k|y_o|\right) - \frac{\pi i}{2k|y_o|}\left[I_1\left(k|y_o|\right) - L_1\left(k|y_o|\right) - \frac{2}{\pi}\right]\right\} \tag{44}$$

Differentiating equation (41) twice with respect to z and setting $z = 0$ gives

$$\left(\frac{\partial^2 F_2''}{\partial z^2}\right)_{z=0} = -\frac{k^2}{l^2}\left[\frac{1}{k^2 y_o^2} + \frac{i}{k}\int_{|y_o|}^{x_o}\frac{e^{\frac{ik}{2}\left(\lambda - \frac{y_o^2}{\lambda}\right)}}{\lambda^2}\,d\lambda\right] \tag{45}$$

After performing an integration by parts and collecting terms, equation (45) may be written as

$$\left(\frac{\partial^2 F_2''}{\partial z^2}\right)_{z=0} = \frac{k^2}{l^2}\left[\frac{1}{k^2 y_o^2} - \frac{2}{k^2 y_o^2}e^{\frac{i}{2}\left(kx_o - \frac{k^2 y_o^2}{kx_o}\right)} + \frac{i}{k^2 y_o^2}\int_{k|y_o|}^{kx_o} e^{\frac{i}{2}\left(\lambda - \frac{k^2 y_o^2}{\lambda}\right)}\,d\lambda\right] \tag{46}$$

Equations (44) and (46) are combined to give $\left(\frac{\partial^2 F_2}{\partial z}\right)_{z=0}$. Then, in accordance with equation (4), there is obtained for $K(x_o,y_o)_{M=1}$:

For $x_o > 0$,

$$K(x_o,y_o)_{M=1} = \frac{k^2}{l^2} e^{-ikx_o}\left\{-\frac{1}{k|y_o|} K_1\left(k|y_o|\right) - \frac{\pi i}{2k|y_o|}\left[I_1\left(k|y_o|\right) - L_1\left(k|y_o|\right) - \frac{2}{\pi}\right] + \right.$$

$$\left. \frac{1}{k^2{y_o}^2} - \frac{2}{k^2{y_o}^2} e^{\frac{i}{2}\left(kx_o - \frac{k^2{y_o}^2}{kx_o}\right)} + \frac{i}{k^2{y_o}^2}\int_{k|y_o|}^{kx_o} e^{\frac{i}{2}\left(\lambda - \frac{k^2{y_o}^2}{\lambda}\right)} d\lambda\right\} \tag{47a}$$

and, for $x_o \leqq 0$,

$$K(x_o,y_o)_{M=1} = 0 \tag{47b}$$

The integral appearing in equation (47a) is finite and proper and can be evaluated by numerical procedures.

Treatment of the Steady and Incompressible Cases

It is of interest to consider the form of the kernel function given in equation (20) for some particular values of M and k. In the following sections a discussion is given for the steady case ($k = 0$) and the incompressible case ($M = 0$). The two-dimensional case is handled in appendix B.

Reduction of the kernel for the case of steady flow.- In order to obtain the reduction of the kernel for the case of steady flow, consider the expanded form given by equation (26). As $k \longrightarrow 0$, there results the following expression

$$K(x_o,y_o)_{k=0} = -\frac{1}{l^2}\left(\frac{1}{{y_o}^2} + \frac{x_o}{{y_o}^2\sqrt{{x_o}^2 + \beta^2{y_o}^2}}\right) \tag{48}$$

which represents the downwash of a pressure doublet for steady flow. This result serves as a partial check as to the correctness of the expression for $K(x_o,y_o)$ given by equation (20).

By replacing y_o in equation (48) by $y - \eta$ and integrating from -1 to 1 with respect to η, there is obtained

$$⨍_{-1}^{1} K(x_o,y_o)d\eta = -\frac{1}{l^2}\left[\frac{x_o + \sqrt{{x_o}^2 + \beta^2(y - 1)^2}}{x_o(y - 1)} - \frac{x_o + \sqrt{{x_o}^2 + \beta^2(y + 1)^2}}{x_o(y + 1)}\right] \tag{49}$$

where the symbol $⨍$ indicates that the principal value or finite part of the improper integral must be taken. (See, for example, ref. 24 for a discussion of finite parts of such integrals.) This result corresponds to the downwash produced by a simple horseshoe vortex two units wide. An equivalent expression for incompressible flow is given, for example, in reference 25, where in contrast to the present notation, x_o has been chosen as positive forward.

Reduction of the kernel for $M = 0$.- In order to effect the reduction of the kernel for the incompressible case, the expressions for F_1, equation (15), and F_2, equation (18), will be examined for the limit $M \longrightarrow 0$:

From equation (15)

$$\lim_{\substack{M\to 0\\ z\to 0}} \frac{\partial^2 F_1}{\partial z^2} = \frac{k}{y_o}\left\{-K_1\left(k|y_o|\right) - \frac{\pi i}{2}\left[I_1\left(k|y_o|\right) - L_1\left(k|y_o|\right)\right] + i\right\} \tag{50}$$

and from equation (18)

$$\lim_{\substack{M\to 0\\ z\to 0}} \frac{\partial^2 F_2}{\partial z^2} = \frac{ik}{|y_o|}\int_0^{\sinh^{-1}\frac{x_o}{|y_o|}} \sinh\theta\, e^{ik|y_o|\sinh\theta}\, d\theta - \frac{x_o}{y_o^2\sqrt{x_o^2 + y_o^2}}\, e^{ikx_o} \tag{51}$$

Integrating by parts yields

$$\lim_{\substack{M\to 0\\ z\to 0}} \frac{\partial^2 F_2}{\partial z^2} = \frac{ik}{y_o^2} - \frac{ik}{y_o} + \frac{k}{y_o^2}\int_0^{x_o} \sqrt{y_o^2 + \lambda^2}\, e^{ik\lambda}\, d\lambda - \frac{x_o}{y_o^2\sqrt{x_o^2 + y_o^2}}\, e^{ikx_o} \tag{52}$$

Combining the results from F_1 and F_2 gives for the kernel function

$$K(x_o,y_o)_{M=0} = \frac{e^{-ikx_o}}{l^2}\left\{-\frac{k}{|y_o|}K_1\left(k|y_o|\right) - \frac{i\pi k}{2|y_o|}\left[I_1\left(k|y_o|\right) - L_1\left(k|y_o|\right)\right] - \frac{x_o}{y_o^2\sqrt{x_o^2 + y_o^2}}e^{ikx_o} + \frac{ik\sqrt{x_o^2 + y_o^2}}{y_o^2}e^{ikx_o} + \frac{k^2}{y_o^2}\int_0^{x_o}\sqrt{\lambda^2 + y_o^2}\,e^{ik\lambda}\,d\lambda\right\} \tag{53}$$

By setting $x_o = 0$ in equation (53), a form is obtained which can be shown to agree with results derived by Küssner for the case $M = 0$, $x_o = 0$ (ref. (26)).

Some Remarks on Evaluation of the Kernel Function

In regard to evaluating the kernel function for specific values of x_o, y_o, M, and k, an approximation for the function

$$\left[K(x_o,y_o) - K'(x_o,y_o)\right]$$

can be obtained by making use of the series expansions for K_1 (eq. (23)) and for $(I_1 - L_1)$ (eq. (24)) and expanding all other terms of $K(x_o,y_o)$ (eq. (20)) into a power series in terms of k. After performing these expansions and collecting terms with respect to powers of k, there is obtained

$$
\begin{aligned}
\left[K(x_o,y_o) - K'(x_o,y_o)\right] \approx \frac{e^{-ikx_o}}{l^2\beta^2}\Bigg\{ & \frac{k^2}{2}\left[M - \frac{x_o}{\sqrt{x_o^2 + \beta^2 y_o^2}} - \beta^2\left(\gamma - \frac{1}{2}\right) - \frac{i\pi\beta^2}{2}\right] + \\
& \frac{ik^3}{6\beta^2}\left[2M^3 x_o + \frac{\left(1 - 3M^2\right)x_o^2 + \left(2 - 3M^2\right)\beta^2 y_o^2}{\sqrt{x_o^2 + \beta^2 y_o^2}}\right] + \\
& \frac{k^4}{192\beta^4}\Bigg[\left(12M\beta^2 - 20M^3\beta^2 + 15\beta^6 - 12\beta^6\gamma\right)y_o^2 - 32M^3 x_o^2 + \\
& \frac{4\left(3M^4 + 6M^2 - 1\right)x_o^3 + 12\beta^2\left(M^4 + 2M^2 - 1\right)x_o y_o^2}{\sqrt{x_o^2 + \beta^2 y_o^2}} - \\
& 12\beta^6 y_o^2 \log\frac{k\left(\sqrt{x_o^2 + \beta^2 y_o^2} - x_o\right)}{2(1 - M)} - i\pi 6\beta^6 y_o^2\Bigg] + \\
& \frac{ik^5}{360\beta^6}\Bigg[\left(15M^4 + 10M^2 - 1\right)x_o^2\sqrt{x_o^2 + \beta^2 y_o^2} - 4M^3\left(5 + M^2\right)x_o^3 + \\
& \frac{3\beta^4 y_o^4}{\sqrt{x_o^2 + \beta^2 y_o^2}} - 12M^5\beta^2 x_o y_o^2 - 5\beta^4\left(3M^2 - 1\right)y_o^2\sqrt{x_o^2 + \beta^2 y_o^2}\Bigg] \cdot \cdot \cdot \Bigg\}
\end{aligned}
\tag{54}
$$

For values of the parameters that satisfy the following inequalities

$$
\left.
\begin{aligned}
& \frac{ky_o}{\beta} < 1 \\
& \frac{k}{\beta^2}\left(x_o - M\sqrt{x_o^2 + \beta^2 y_o^2}\right) < 1
\end{aligned}
\right\}
\tag{55}
$$

equation (54) yields results that are correct to within 2 percent. For

values of the variables x_o, y_o, M, and k that do not satisfy the inequalities (55), values of the function $\left[K(x_o,y_o) - K'(x_o,y_o)\right]$ are perhaps most easily and economically obtained by numerically evaluating the two integrals in equation (20) and making use of available tables to evaluate other terms of $\left[K(x_o,y_o) - K'(x_o,y_o)\right]$.

It is pointed out that extensive tables of the Bessel functions K_1 and I_1 may be found in reference 27. A table of the Struve function L_1 with second and fourth differences for interpolation purposes may be found in reference 28.

CONCLUDING REMARKS

The main purpose of this paper was to present the kernel function of the integral equation relating the downwash to the lift distribution in a form that can be computed. This purpose has been achieved by the presentation of the kernel in a form given in equation (20). This equation has been converted to a form more suitable for calculation by isolating the singularities as shown in equations (30) and (31). The special case of $M = 1$ is given in equations (47). The forms of the kernel function for other limiting cases, namely $k = 0$ and $M = 0$, are given in equations (48) and (53), respectively.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., September 18, 1953.

APPENDIX A

DERIVATION OF THE INTEGRAL EQUATION THAT RELATES THE DOWNWASH AND LIFT FOR A FINITE WING BASED ON REFERENCE 21

In keeping with the concepts of linear theory the wing is considered as a nearly plane, impenetrable surface. Let this surface S lie nearly in the xy-plane, as indicated in sketch 1 of the body of the paper, and let it and the x,y,z coordinate system to which it is referred be assumed to move at a uniform speed V in the negative x-direction. At the same time let each point of the wing be assumed to undergo harmonic translations of small amplitude $Z_m(x,y,t)$ at circular frequency ω and let c represent velocity of sound in the medium.

The problem for an oscillating wing consists in solving the wave equation subject to certain boundary conditions. The wave equation in rectangular coordinates is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{c^2}\left(V\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^2\psi = 0 \tag{A1}$$

The independent variable ψ in equation (A1) is regarded herein as an acceleration potential; as such it is directly proportional to a perturbation pressure field and is related to a velocity potential ϕ as follows:

$$\psi = \frac{\partial\phi}{\partial t} + V\frac{\partial\phi}{\partial x} \tag{A2}$$

In order to complete the boundary-value problem for the wing, it is desirable to calculate the downwash $w(x,y,z,t) = \frac{\partial\phi}{\partial z}$ associated with ψ. Assuming this downwash to be harmonic with regard to time implies that both potentials ϕ and ψ are harmonic with regard to time and can be written, therefore, as

$$\left.\begin{aligned}\phi(x,y,z,t) &= e^{i\omega t}\,\bar{\phi}(x,y,z)\\ \psi(x,y,z,t) &= e^{i\omega t}\,\bar{\psi}(x,y,z)\end{aligned}\right\} \tag{A3}$$

With these expressions for ϕ and ψ, equation (A2) becomes independent of time and reduces to an ordinary equation with one dependent variable, namely

$$\bar{\psi} = i\omega\bar{\phi} + V\frac{d\bar{\phi}}{dx} \tag{A4}$$

This equation can be integrated with respect to x to give

$$\bar{\phi} = \frac{e^{-\frac{i\omega x}{V}}}{V}\int_{-\infty}^{x}\bar{\psi}(\lambda,y,z)e^{\frac{i\omega\lambda}{V}}d\lambda \tag{A5}$$

where the lower limit of integration is chosen, for later convenience, so as to satisfy the condition that ϕ vanish as $x \longrightarrow -\infty$.

The boundary-value problem for the wing may now be expressed mathematically as follows: Under the assumption of harmonic motion the differential equation, equation (A1), becomes

$$\frac{\partial^2\bar{\psi}}{\partial x^2} + \frac{\partial^2\bar{\psi}}{\partial y^2} + \frac{\partial^2\bar{\psi}}{\partial z^2} - \frac{1}{c^2}\left(V\frac{\partial}{\partial x} + i\omega\right)^2\bar{\psi} = 0 \tag{A6}$$

In order to insure tangential flow at the wing surface, the potential must satisfy the downwash condition

$$\bar{w}(x,y) = \left(\frac{\partial\bar{\phi}}{\partial z}\right)_{z=0} = \left(V\frac{\partial}{\partial x} + i\omega\right)\bar{Z}_m(x,y) \tag{A7}$$

where $\bar{w}$ and $\bar{Z}_m$ are amplitudes of velocity and displacements, respectively, and are assumed to be known from the motion of the wing. At $z = 0$, the pressure

$$p = -\rho(\psi)_{z=0} \tag{A8}$$

must be zero at all points (x,y) off the wing. At all points on the wing ψ is allowed to be discontinuous and the value of p at a given point is determined by the magnitude of the discontinuity in ψ at the point. In the neighborhood of the trailing edge, p must go to zero, corresponding to the Kutta condition.

One other condition, that ϕ vanish far ahead of the wing, is inherently satisfied by the relation between ϕ and ψ given in equation (A5).

The potential ψ_o at point (x,y,z) due to a harmonically pulsating doublet located in the xy-plane at $(\xi,\eta,0)$ that satisfies equation (A6) is

$$\psi_o = A\frac{\partial}{\partial z}\frac{e^{i\omega\left[t+\frac{M}{c\beta^2}(x-\xi)-\frac{R'}{c\beta^2}\right]}}{R'} \tag{A9}$$

where

$$R' = \sqrt{(x-\xi)^2 + \beta^2(y-\eta)^2 + \beta^2 z^2}$$

and the factor A is a strength and dimensionality factor that makes possible different uses and interpretations of the potential ψ_o. If ψ_o is considered as an acceleration potential and substituted into equation (A5), there is obtained a corresponding velocity potential ϕ_o which may be written as

$$\phi_o = A\frac{\partial}{\partial z}e^{-\frac{i\omega(x-\xi)}{V}}\int_{-\infty}^{\lambda=x-\xi}\frac{e^{i\omega\left(t+\frac{\lambda}{V}+\frac{M\lambda}{c\beta^2}-\frac{R}{c\beta^2}\right)}}{R}\,d\lambda \tag{A10}$$

where

$$R = \sqrt{\lambda^2 + \beta^2(y-\eta)^2 + \beta^2 z^2}$$

The downwash $\frac{\partial\phi_o}{\partial z}$ associated with ψ_o may be written as

$$\frac{\partial\phi_o}{\partial z} = A\frac{\partial^2}{\partial z^2}e^{-\frac{i\omega x_o}{V}}\int_{-\infty}^{x_o}\frac{e^{i\bar{\omega}\left(\lambda-M\sqrt{\lambda^2+r^2}\right)}}{\sqrt{\lambda^2+r^2}}\,d\lambda \tag{A11}$$

where $x_0 = x - \xi$, $\bar{\omega} = \omega/V\beta^2$, and $r = \beta\sqrt{(y - \eta)^2 + z^2}$. With the use of this equation and the concept of solving linear boundary-value problems by means of superposition of elementary solutions to the governing differential equation, the boundary-value problem under discussion can be written as an integral equation, namely

$$\bar{w}(x,y) = \lim_{z \to 0} A \iint_S L(\xi,\eta) e^{-\frac{i\omega x_0}{V}} d\xi \, d\eta \, \frac{\partial^2}{\partial z^2} \int_{-\infty}^{x_0} \frac{e^{i\bar{\omega}\left(\lambda - M\sqrt{\lambda^2 + r^2}\right)}}{\sqrt{\lambda^2 + r^2}} d\lambda \qquad \text{(A12)}$$

where S represents the surface of the wing and $L(\xi,\eta)$ represents an unknown lift distribution or doublet strength on S. Equation (A12) may be seen to correspond essentially to equations (1) and (2).

If the distribution function $L(\xi,\eta)$ in equation (A12) is determined in accordance with the boundary conditions discussed in the preceding paragraph, equation (A12) can be considered as a complete solution to the boundary-value problem for an oscillating finite wing in compressible flow. It is also to be noted that equation (A12) can be considered to represent a solution to the so-called "indirect" problem, that is, that of finding the downwash distribution associated with a given lift distribution.

APPENDIX B

REDUCTION OF THE KERNEL FUNCTION FOR THREE-DIMENSIONAL FLOW TO THAT FOR TWO-DIMENSIONAL FLOW

The purpose of this appendix is to show that integration of the kernel function $K(x_o,y_o)$ from $-\infty$ to $+\infty$ with respect to $\eta = y - y_o$ leads to a known result for two-dimensional flow. The kernel is first modified to a form that, for the present case, is easier to handle. Then, after performing an integration by parts on the modified kernel, the form of the kernel for the two-dimensional case is given (eq. (B18)). In addition, the special cases of $M = 1$ (eq. (B23)) and $M = 0$ (eq. (B30)) are also shown.

The integration under consideration with respect to η is equivalent to an integration with respect to y_o, namely

$$l \int_{-\infty}^{\infty} K(x_o,y-\eta)\, d\eta = l \int_{-\infty}^{\infty} K(x_o,y_o)\, dy_o \tag{B1}$$

It is remarked in advance that since z has been made zero in the expression for $K(x_o,y_o)$, equation (20), it is necessary to employ the concept of "finite parts of infinite integrals" when integrating this function across the singularities at $y_o = 0$. Use of this concept gives the same results that could be obtained by the more arduous task of performing the integrations before setting z equal to zero.

<u>Modification of the kernel</u>.- In order to effect the desired modification of the expression for $K(x_o,y_o)$ given by equation (20), consider the first integral of the expression, namely

$$-k^2 \int_0^{M/\beta} \sqrt{1 + \tau^2}\, e^{-ik|y_o|\tau}\, d\tau \tag{B2}$$

This integral can be written as

$$\lim_{\delta \to 0} -k^2 \int_0^{\infty} \sqrt{1 + \tau^2}\, e^{-\tau(\delta+ik|y_o|)}\, d\tau + k^2 \int_{M/\beta}^{\infty} \sqrt{1 + \tau^2}\, e^{-ik|y_o|\tau}\, d\tau \tag{B3}$$

but according to page 331 of reference 22

$$\int_0^\infty \sqrt{1+\tau^2}\, e^{-\tau\left(\delta+ik|y_o|\right)} d\tau = \frac{\pi}{2\left(\delta+ik|y_o|\right)}\left[H_1\left(\delta+ik|y_o|\right) - Y_1\left(\delta+ik|y_o|\right)\right] \tag{B4}$$

where H_1 is the unmodified Struve function of first order and Y_1 is the Bessel function of the second kind of first order. In the limit as $\delta \longrightarrow 0$ these expressions have the following values:

For the first expression in the bracket (see ref. 22, p. 329)

$$\lim_{\delta \to 0} H_1\left(\delta + ik|y_o|\right) = H_1\left(ik|y_o|\right) = -L_1\left(k|y_o|\right) \tag{B5}$$

and for the second expression (see ref. 22, pp. 77 and 78)

$$\begin{aligned}\lim_{\delta \to 0} Y_1\left(\delta + ik|y_o|\right) &= -iH_1^{(1)}\left(ik|y_o|\right) + iJ_1\left(ik|y_o|\right) \\ &= \frac{2i}{\pi} K_1\left(k|y_o|\right) - I_1\left(k|y_o|\right)\end{aligned} \tag{B6}$$

where $H_1^{(1)}$ denotes the Hankel function of the first kind of first order. With the use of equations (B3) to (B6), expression (B2) can be written as

$$\begin{aligned}-k^2\int_0^{M/\beta} \sqrt{1+\tau^2}\, e^{-ik|y_o|\tau}\, d\tau = k^2\int_{M/\beta}^\infty \sqrt{1+\tau^2}\, e^{-ik|y_o|\tau}\, d\tau \, + \\ \frac{k}{|y_o|}\left\{K_1\left(k|y_o|\right) + \frac{\pi i}{2}\left[I_1\left(k|y_o|\right) - L_1\left(k|y_o|\right)\right]\right\}\end{aligned} \tag{B7}$$

Substituting this result into equation (20) of the text gives the modified form of $K(x_o,y_o)$ sought, namely

$$K(x_o,y_o) = \frac{e^{-ikx_o}}{l^2}\left[\frac{ik}{\beta|y_o|} e^{-\frac{iMk|y_o|}{\beta}} + \frac{1}{My_o^2} e^{-\frac{iMk|y_o|}{\beta}} - \right.$$

$$\frac{Mx_o + \sqrt{x_o^2 + \beta^2 y_o^2}}{My_o^2\sqrt{x_o^2 + \beta^2 y_o^2}} e^{\frac{ik}{\beta^2}\left(x_o - M\sqrt{x_o^2+\beta^2 y_o^2}\right)} +$$

$$\left. k^2 \int_{M/\beta}^{\infty} \sqrt{1 + \tau^2}\, e^{-ik|y_o|\tau}\, d\tau + \frac{ik}{My_o^2}\int_0^{x_o} e^{\frac{ik}{\beta^2}\left(\lambda - M\sqrt{\lambda^2+\beta^2 y_o^2}\right)} d\lambda \right]$$

(B8)

<u>Integration of modified kernel</u>.- Since the expression for $K(x_o,y_o)$ is symmetrical with respect to y_o, that is, $K(x_o,-y_o) = K(x_o,+y_o)$, the integration under consideration can be expressed as

$$l\int_{-\infty}^{\infty} K\left(x_o,|y_o|\right) dy_o = 2l\int_0^{\infty} K(x_o,y_o)\, dy_o \qquad \text{(B9)}$$

where, on the right, the absolute-value signs on y_o can be dropped.

After performing an integration by parts by letting

$$dv = 2\,\frac{e^{-ikx_o}}{l}\,\frac{dy_o}{y_o^2} \quad ; \quad v = -2\,\frac{e^{-ikx_o}}{l}\,\frac{1}{y_o} \qquad \text{(B10)}$$

and

$$u = \frac{iky_o}{\beta} e^{-\frac{iMky_o}{\beta}} + \frac{1}{M} e^{-\frac{iMky_o}{\beta}} - \frac{Mx_o + \sqrt{x_o^2 + \beta^2 y_o^2}}{M\sqrt{x_o^2 + \beta^2 y_o^2}} e^{\frac{ik}{\beta^2}\left(x_o - M\sqrt{x_o^2+\beta^2 y_o^2}\right)} +$$

$$k^2 y_o^2 \int_{M/\beta}^{\infty} \sqrt{1 + \tau^2}\, e^{-iky_o\tau}\, d\tau + \frac{ik}{M}\int_0^{x_o} e^{\frac{ik}{\beta^2}\left(\lambda - M\sqrt{\lambda^2+\beta^2 y_o^2}\right)} d\lambda \qquad \text{(B11)}$$

or

$$du = \left\{\left[\frac{\beta^2 y_o x_o}{\left(x_o^2 + \beta^2 y_o^2\right)^{3/2}} + \frac{ikMx_o y_o}{x_o^2 + \beta^2 y_o^2} + \frac{iky_o}{\sqrt{x_o^2 + \beta^2 y_o^2}}\right] e^{\frac{ik}{\beta^2}\left(x_o - M\sqrt{x_o^2+\beta^2 y_o^2}\right)} + \right.$$

$$\left. k^2 y_o \int_{M/\beta}^{\infty} \frac{e^{-iky_o\tau}}{\sqrt{1+\tau^2}} d\tau + k^2 y_o \int_0^{x_o} \frac{e^{\frac{ik}{\beta^2}\left(\lambda - M\sqrt{\lambda^2+\beta^2 y_o^2}\right)}}{\sqrt{\lambda^2 + \beta^2 y_o^2}} d\lambda \right\} dy_o \qquad \text{(B12)}$$

There is obtained for $\left.uv\right]_0^{\infty}$

$$\left.uv\right]_0^{\infty} = 2\,\frac{e^{-ikx_o}}{l}\left\{-\frac{1}{y_o}\left[\frac{iky_o}{\beta} e^{-\frac{iMky_o}{\beta}} + \frac{1}{M} e^{-\frac{iMky_o}{\beta}} - \right.\right.$$

$$\left(\frac{x_o}{\sqrt{x_o^2 + \beta^2 y_o^2}} + \frac{1}{M}\right) e^{\frac{ik}{\beta^2}\left(x_o - M\sqrt{x_o^2+\beta^2 y_o^2}\right)} + k^2 y_o^2 \int_{M/\beta}^{\infty} \sqrt{1 + \tau^2}\, e^{-iky_o\tau}\, d\tau +$$

$$\left.\left.\frac{ik}{M}\int_0^{x_o} e^{\frac{ik}{\beta^2}\left(\lambda - M\sqrt{\lambda^2+\beta^2 y_o^2}\right)} d\lambda \right]_{y_o=0}^{y_o=\infty}\right\} \qquad \text{(B13)}$$

This expression vanishes at its upper limit $y_o = \infty$ and is singular at its lower limit $y_o = 0$. However, by not making $z \longrightarrow 0$ in the derivation of $K(x_o,y_o)$ until after this stage is reached, this singular value is canceled by other terms that have otherwise been dropped. Thus, the expression (B13) may be considered to be zero, which is the value of its finite part. The integration under consideration is then reduced to the value of $-\int_0^\infty v\,du$ which is

$$-\int_0^\infty v\,du = 2\,\frac{e^{-ikx_o}}{l}\int_0^\infty\left\{\left[\frac{\beta^2 x_o}{\left(x_o^2+\beta^2y_o^2\right)^{3/2}}+\frac{ikMx_o}{x_o^2+\beta^2y_o^2}+\frac{ik}{\sqrt{x_o^2+\beta^2y_o^2}}\right]e^{\frac{ik}{\beta^2}\left(x_o-M\sqrt{x_o^2+\beta^2y_o^2}\right)}+k^2\int_{M/\beta}^\infty\frac{e^{-iky_o\tau}}{\sqrt{1+\tau^2}}\,d\tau+k^2\int_0^{x_o}\frac{e^{\frac{ik}{\beta^2}\left(\lambda-M\sqrt{\lambda^2+\beta^2y_o^2}\right)}}{\sqrt{\lambda^2+\beta^2y_o^2}}\,d\lambda\right\}dy_o$$

$$=2\,\frac{e^{-ikx_o}}{l}\int_0^\infty\left[\left(\frac{ikM}{x_o}+\frac{ik}{\sqrt{x_o^2+\beta^2y_o^2}}\right)e^{\frac{ik}{\beta^2}\left(x_o-M\sqrt{x_o^2+\beta^2y_o^2}\right)}+k^2\int_{M/\beta}^\infty\frac{e^{-iky_o\tau}}{\sqrt{1+\tau^2}}\,d\tau+k^2\int_0^{x_o}\frac{e^{\frac{ik}{\beta^2}\left(\lambda-M\sqrt{\lambda^2+\beta^2y_o^2}\right)}}{\sqrt{\lambda^2+\beta^2y_o^2}}\,d\lambda\right]dy_o \qquad \text{(B14)}$$

The terms of this expression are treated separately in the next three equations:

First (see ref. 22, p. 180)

$$2\int_0^\infty\left(\frac{ikM}{x_o}+\frac{ik}{\sqrt{x_o^2+\beta^2y_o^2}}\right)e^{\frac{ik}{\beta^2}\left(x_o-M\sqrt{x_o^2+\beta^2y_o^2}\right)}dy_o = \frac{2ik}{\beta}e^{\frac{ikx_o}{\beta^2}}\int_0^\infty\left(\frac{|x_o|}{x_o}M\cosh\theta+1\right)e^{-\frac{ikM}{\beta^2}|x_o|\cosh\theta}d\theta$$

$$= -\frac{i\pi k}{\beta}e^{\frac{ikx_o}{\beta^2}}\left[M\frac{|x_o|}{x_o}H_1^{(2)}\left(\frac{kM|x_o|}{\beta^2}\right)+iH_o^{(2)}\left(\frac{kM|x_o|}{\beta^2}\right)\right] \tag{B15}$$

second

$$2k^2\int_0^\infty dy_o\int_{M/\beta}^\infty\frac{e^{-iky_o\tau}}{\sqrt{1+\tau^2}}d\tau = 2k^2\int_{M/\beta}^\infty\frac{d\tau}{\sqrt{1+\tau^2}}\int_0^\infty e^{-iky_o\tau}dy_o$$

$$= -2ik\int_{M/\beta}^\infty\frac{d\tau}{\tau\sqrt{1+\tau^2}}$$

$$= -2ik\log\frac{1+\beta}{M} \tag{B16}$$

and third (see ref. 22, p. 180)

$$2k^2\int_0^\infty dy_o\int_0^{x_o}\frac{e^{\frac{ik}{\beta^2}\left(\lambda-M\sqrt{\lambda^2+\beta^2y_o^2}\right)}}{\sqrt{\lambda^2+\beta^2y_o^2}}d\lambda = 2k^2\int_0^{x_o}e^{\frac{ik\lambda}{\beta^2}}d\lambda\int_0^\infty\frac{e^{-\frac{ik}{\beta^2}M\sqrt{\lambda^2+\beta^2y_o^2}}}{\sqrt{\lambda^2+\beta^2y_o^2}}dy_o$$

$$= \frac{2k^2}{\beta}\int_0^{x_o}e^{\frac{ik\lambda}{\beta^2}}d\lambda\int_0^\infty e^{-\frac{ikM}{\beta^2}|\lambda|\cosh\theta}d\theta$$

$$= -\frac{\pi ik^2}{\beta}\int_0^{x_o}e^{\frac{ik\lambda}{\beta^2}}H_o^{(2)}\left(\frac{kM}{\beta^2}|\lambda|\right)d\lambda \tag{B17}$$

Substituting the results in equations (B15) to (B17) into equation (B14) gives

$$l\int_{-\infty}^{\infty} K(x_o,y_o)\,dy_o = -\frac{\pi k}{l\beta}\,e^{-ikx_o}\left\{e^{\frac{ikx_o}{\beta^2}}\left[\frac{iM|x_o|}{x_o}\,H_1^{(2)}\left(\frac{kM|x_o|}{\beta^2}\right) - H_0^{(2)}\left(\frac{kM|x_o|}{\beta^2}\right)\right] + \frac{2i}{\pi}\,\beta\log\frac{1+\beta}{M} + ik\int_0^{x_o} e^{\frac{ik\lambda}{\beta^2}}\,H_0^{(2)}\left(\frac{kM|\lambda|}{\beta^2}\right)d\lambda\right\} \tag{B18}$$

This result is a form of the expression for the kernel function of Possio's integral equation relating pressure and downwash for a two-dimensional oscillating wing in subsonic compressible flow. It checks the results given, for example, in reference 29.

Reduction of kernel for $M = 1$.- The kernel function for $M = 1$ may be written as (see eq. (47a))

$$K(x_o,y_o)_{M=1} = \frac{k^2}{l^2}\,e^{-ikx_o}\left\{-\frac{1}{k|y_o|}\,K_1(k|y_o|) - \frac{\pi i}{2k|y_o|}\left[I_1(k|y_o|) - L_1(k|y_o|) - \frac{2}{\pi}\right] + \frac{1}{k^2y_o^2} - \frac{2}{k^2y_o^2}\,e^{\frac{ik}{2}\left(x_o - \frac{y_o^2}{x_o}\right)} + \frac{i}{k^2y_o^2}\int_0^{kx_o} e^{\frac{i}{2}\left(\lambda - \frac{k^2y_o^2}{\lambda}\right)}d\lambda - \frac{i}{k^2y_o^2}\int_0^{k|y_o|} e^{\frac{i}{2}\left(\lambda - \frac{k^2y_o^2}{\lambda}\right)}d\lambda\right\} \tag{B19}$$

The second integral appearing in this equation can be shown to cancel several of the terms so that the kernel becomes

$$K(x_o,y_o)_{M=1} = -\frac{e^{-ikx_o}}{l^2}\left[\frac{2}{y_o^2}\,e^{\frac{ik}{2}\left(x_o - \frac{y_o^2}{x_o}\right)} - \frac{i}{y_o^2}\int_0^{kx_o} e^{\frac{i}{2}\left(\lambda - \frac{k^2y_o^2}{\lambda}\right)}d\lambda\right] \tag{B20}$$

so that the kernel for the sonic case in two-dimensional flow may be written as

$$l\int_{-\infty}^{\infty} K(x_o,y_o)_{M=1}\, dy_o = -\frac{e^{-ikx_o}}{l}\left(2e^{\frac{ikx_o}{2}}\int_{-\infty}^{\infty}\frac{e^{-\frac{iky_o^2}{2x_o}}}{y_o^2}\, dy_o - i\int_0^{kx_o} e^{\frac{i\lambda}{2}}\, d\lambda\int_{-\infty}^{\infty}\frac{e^{-\frac{ik^2y_o^2}{2\lambda}}}{y_o^2}\, dy_o\right) \tag{B21}$$

Integrating equation (B21) by parts with respect to y_o, retaining only finite parts of the integrated results, and making use of the relation $i\int_{-\infty}^{\infty} e^{-ia^2\tau^2}\, d\tau = 2i\int_0^{\infty} e^{-ia^2\tau^2}\, d\tau = \frac{\sqrt{\pi i}}{a}$ yields

$$l\int_{-\infty}^{\infty} K(x_o,y_o)_{M=1}\, dy_o = -\frac{e^{-ikx_o}}{l}\left\{2e^{\frac{ikx_o}{2}}\left(-\frac{1}{y_o}e^{-\frac{iky_o^2}{2x_o}}\right]_{-\infty}^{\infty} - \frac{ik}{x_o}\int_{-\infty}^{\infty} e^{-\frac{iky_o^2}{2x_o}}\, dy_o\right) - i\int_0^{kx_o} e^{\frac{i\lambda}{2}}\, d\lambda\left(-\frac{1}{y_o}e^{-\frac{ik^2y_o^2}{2\lambda}}\right]_{-\infty}^{\infty} - \frac{ik^2}{\lambda}\int_{-\infty}^{\infty} e^{-\frac{ik^2y_o^2}{2\lambda}}\, dy_o\right)\right\}$$

$$= -\frac{2k\sqrt{\pi}}{l}e^{-ikx_o}\left(-2\sqrt{\frac{i}{2kx_o}}\, e^{\frac{ikx_o}{2}} + i\sqrt{\frac{i}{2}}\int_0^{kx_o}\frac{e^{\frac{i\lambda}{2}}}{\sqrt{\lambda}}\, d\lambda\right) \tag{B22}$$

Finally, the kernel for the sonic case in two-dimensional flow may be written as

$$l\int_{-\infty}^{\infty} K(x_o,y_o)_{M=1}\, dy_o = \frac{4}{l}\sqrt{\frac{\pi i}{2}}\, e^{-ikx_o}\left(\frac{k}{\sqrt{kx_o}}\, e^{\frac{ikx_o}{2}} - ik\sqrt{\pi}\int_0^{\sqrt{\frac{kx_o}{\pi}}} e^{\frac{i\pi\lambda^2}{2}}\, d\lambda\right) \tag{B23}$$

It may be noted that the integrals in this equation are readily expressible in terms of Fresnel integrals

$$C(x) = \int_0^x \cos\frac{\pi}{2}t^2\, dt$$

and

$$S(x) = \int_0^x \sin\frac{\pi}{2}t^2\, dt$$

Reduction of kernel for $M = 0$.- For $M = 0$ it is convenient to modify the kernel function before integrating with respect to y_o. For this purpose use is made of the relation (see eq. (B7)):

$$-\frac{k}{|y_o|} K_1\left(k|y_o|\right) - \frac{i\pi k}{2|y_o|}\left[I_1\left(k|y_o|\right) - L_1\left(k|y_o|\right)\right] = k^2\int_0^{\infty} \sqrt{1 + \tau^2}\, e^{-ik|y_o|\tau}\, d\tau$$

$$= \frac{k^2}{{y_o}^2}\int_0^{\infty} \sqrt{{y_o}^2 + \lambda^2}\, e^{-ik\lambda}\, d\lambda \qquad \text{(B24)}$$

and the relation

$$\frac{k^2}{{y_o}^2}\int_0^{x_o} \sqrt{\lambda^2 + {y_o}^2}\, e^{ik\lambda}\, d\lambda = \frac{k^2}{{y_o}^2}\int_{-x_o}^{0} \sqrt{{y_o}^2 + \lambda^2}\, e^{-ik\lambda}\, d\lambda \qquad \text{(B25)}$$

With these relations the expression for $K(x_o,y_o)_{M=0}$, equation (53), can be written as

$$K(x_o,y_o)_{M=0} = \frac{e^{-ikx_o}}{l^2}\left(-\frac{x_o}{{y_o}^2\sqrt{{x_o}^2 + {y_o}^2}}\, e^{-ikx_o} + \frac{ik\sqrt{{x_o}^2 + {y_o}^2}}{{y_o}^2}\, e^{ikx_o} + \right.$$

$$\left.\frac{k^2}{{y_o}^2}\int_{-x_o}^{\infty} \sqrt{{y_o}^2 + \lambda^2}\, e^{-ik\lambda}\, d\lambda\right) \qquad \text{(B26)}$$

But

$$\frac{k^2}{{y_o}^2}\int_{-x_o}^{\infty} \sqrt{{y_o}^2 + \lambda^2}\, e^{-ik\lambda}\, d\lambda = -\frac{ik\sqrt{{x_o}^2 + {y_o}^2}}{{y_o}^2}\, e^{ikx_o} - \frac{ik}{{y_o}^2}\int_{-x_o}^{\infty} \frac{\lambda}{\sqrt{{y_o}^2 + \lambda^2}}\, e^{-ik\lambda}\, d\lambda$$

$$= -\frac{ik\sqrt{{x_o}^2 + {y_o}^2}}{{y_o}^2}\, e^{ikx_o} + \frac{x_o}{{y_o}^2\sqrt{{x_o}^2 + {y_o}^2}}\, e^{ikx_o} - \int_{-x_o}^{\infty} \frac{e^{-ik\lambda}}{\left({y_o}^2 + \lambda^2\right)^{3/2}}\, d\lambda$$

(B27)

therefore,

$$K(x_o,y_o)_{M=0} = -\frac{e^{-ikx_o}}{l^2}\int_{-x_o}^{\infty}\frac{e^{-ik\lambda}}{\left(y_o^2+\lambda^2\right)^{3/2}}\,d\lambda \tag{B28}$$

Integrating with respect to y_o gives

$$l\int_{-\infty}^{\infty}K(x_o,y_o)_{M=0}\,dy_o = -\frac{2}{l}e^{-ikx_o}\int_0^{\infty}\int_{-x_o}^{\infty}\frac{e^{-ik\lambda}}{\left(y_o^2+\lambda^2\right)^{3/2}}\,d\lambda\,dy_o$$

$$= -\frac{2}{l}e^{-ikx_o}\int_{-x_o}^{\infty}e^{-ik\lambda}\,d\lambda\int_0^{\infty}\frac{dy_o}{\left(y_o^2+\lambda^2\right)^{3/2}}$$

$$= -\frac{2}{l}e^{-ikx_o}\int_{-x_o}^{\infty}\frac{e^{-ik\lambda}}{\lambda^2}\,d\lambda$$

$$= -\frac{2}{l}e^{-ikx_o}\left(\int_{-\infty}^{\infty}\frac{e^{-ik\lambda}}{\lambda^2}\,d\lambda - \int_{x_o}^{\infty}\frac{e^{ik\lambda}}{\lambda^2}\,d\lambda\right) \tag{B29}$$

Integrating each integral in equation (B29) and retaining only finite parts yields

$$l\int_{-\infty}^{\infty}K(x_o,y_o)_{M=0}\,dy_o = -\frac{2}{l}e^{-ikx_o}\left(-\frac{e^{ikx_o}}{x_o} - ik\int_{-\infty}^{\infty}\frac{e^{-ik\lambda}}{\lambda}\,d\lambda - ik\int_{x_o}^{\infty}\frac{e^{ik\lambda}}{\lambda}\,d\lambda\right)$$

$$= -\frac{2}{l}e^{-ikx_o}\left(-\frac{e^{-ikx_o}}{x_o} - 2k\int_0^{\infty}\frac{\sin k\lambda}{\lambda}\,d\lambda - \right.$$

$$\left. ik\int_{x_o}^{\infty}\frac{\cos k\lambda}{\lambda}\,d\lambda + k\int_{x_o}^{\infty}\frac{\sin k\lambda}{\lambda}\,d\lambda\right)$$

$$= -\frac{4\pi k}{l}\left(-\frac{1}{2\pi kx_o} + \frac{1}{2\pi}e^{-ikx_o}\left\{Ci(kx_o) + i\left[Si(kx_o) + \frac{\pi}{2}\right]\right\}\right) \tag{B30}$$

where $Ci(kx_o)$ and $Si(kx_o)$ denote, respectively, the "cosine integral" and "sine integral" functions defined as follows:

$$Ci(x) = -\int_x^\infty \frac{\cos t}{t}\,dt$$

$$Si(x) = \frac{\pi}{2} - \int_x^\infty \frac{\sin t}{t}\,dt$$

The results in the braces of equation (B30) check with results given for this case in reference 14.

REFERENCES

1. Theodorsen, Theodore: General Theory of Aerodynamic Instability and the Mechanism of Flutter. NACA Rep. 496, 1935.

2. Schade, Th., and Krienes, K.: The Oscillating Circular Airfoil on the Basis of Potential Theory. NACA TM 1098, 1947.

3. Kochin, N. E.: Steady Vibrations of Wing of Circular Plan Form. Theory of Wing of Circular Plan Form. NACA TM 1324, 1953.

4. Cicala, P.: Comparison of Theory With Experiment in the Phenomenon of Wing Flutter. NACA TM 887, 1939. (From L'Aerotecnica, vol. 18, no. 4, Apr. 1938, pp. 412-433.)

5. Jones, W. P., and Skan, Sylvia W.: Calculations of Derivatives for Rectangular Wings of Finite Span by Cicala's Method. R. & M. No. 1920, British A.R.C., 1940.

6. Dingel, [M.], and Küssner, [H. G.]: Beiträge zur instationären Tragflächentheorie. VIII.- Die schwingende Tragfläche grosser Streckung. FB Nr. 1774, Deutsche Luftfahrtforschung (Berlin-Adlershof), 1943. (Also available as AAF Translation No. F-TS-935-RE, Air Materiel Command, May 1947 and Library Translation No. 210, British R.A.E., June 1948.)

7. Reissner, Eric: Effect of Finite Span on the Airload Distributions for Oscillating Wings. I - Aerodynamic Theory of Oscillating Wings of Finite Span. NACA TN 1194, 1947.

8. Reissner, Eric, and Stevens, John E.: Effect of Finite Span on the Airload Distributions for Oscillating Wings. II - Methods of Calculation and Examples of Application. NACA TN 1195, 1947.

9. Jones, Robert T.: The Unsteady Lift of a Finite Wing. NACA TN 682, 1939.

10. Biot, M. A., and Boehnlein, C. T.: Aerodynamic Theory of the Oscillating Wing of Finite Span. GALCIT Rep. No. 5, Sept. 1942.

11. Wasserman, L. S.: Aspect Ratio Corrections in Flutter Calculations. MR No. MCREXA5-4595-8-5, Air Materiel Command, Eng. Div., U. S. Air Force, Aug. 26, 1948.

12. Lawrence, H. R., and Gerber, E. H.: The Aerodynamic Forces on Low Aspect Ratio Wings Oscillating in an Incompressible Flow. Jour. Aero. Sci., vol. 19, no. 11, Nov. 1952, pp. 769-781. (Errata issued, vol. 20, no. 4, Apr. 1953, p. 296.)

13. Possio, Camillo: L'Azione aerodinamica sul profilo oscillante in un fluido compressibile a velocità iposonora. L'Aerotecnica, vol. XVIII, fasc. 4, Apr. 1938, pp. 441-458. (Available as British Air Ministry Translation No. 830.)

14. Schwarz, [L.]: Tables for the Calculation of Air Forces of the Vibrating Wing in Compressible Plane Subsonic Flow. AAF Translation No. F-TS-599-RE, Air Materiel Command, Aug. 1946.

15. Dietze, [F.]: The Air Forces of the Harmonically Vibrating Wing in Compressible Medium at Subsonic Velocity (Plane Problem). AAF Translation No. F-TS-506-RE, Air Materiel Command, Nov. 1946.

16. Schade, [Th.]: Numerische Lösung der Possioschen Integralgleichung der schwingenden Tragfläche in ebener Unterschallströmung. I.- Analylischer Teil. UM Nr. 3209, Deutsche Luftfahrtforschung (Berlin-Adlershof), 1944.

17. Haskind, M. D.: Oscillations of a Wing in a Subsonic Gas Flow. Translation No. A9-T-22, Air Materiel Command and Brown Univ. (Contract w33-038-ac-15004(16351)). (From Prikl. Mat. i Mekh. (Moscow), vol. XI, no. 1, 1947, pp. 129-146.)

18. Timman, R., Van de Vooren, A. I., and Greidanus, J. H.: Aerodynamic Coefficients of an Oscillating Airfoil in Two-Dimensional Subsonic Flow. Jour. Aero. Sci., vol. 18, no. 12, Dec. 1951, pp. 797-802.

19. Reissner, Eric: On the Application of Mathieu Functions in the Theory of Subsonic Compressible Flow Past Oscillating Airfoils. NACA TN 2363, 1951.

20. Fettis, Henry E.: Regarding the Computation of Unsteady Air Forces by Means of Mathieu Functions. Jour. Aero. Sci. (Readers' Forum), vol. 20, no. 6, June 1953, pp. 437-438.

21. Küssner, H. G.: General Airfoil Theory. NACA TM 979, 1941. (From Luftfahrtforschung, Bd. 17, Lfg. 11/12, Dec. 10, 1940, pp. 370-378.)

22. Watson, G. N.: A Treatise on the Theory of Bessel Functions. Second ed., The Macmillan Co., 1948.

23. Churchill, Ruel V.: Modern Operational Mathematics in Engineering. McGraw-Hill Book Co., Inc., 1944.

24. Mangler, K. W.: Improper Integrals in Theoretical Aerodynamics. Rep. No. Aero. 2424, British R.A.E., June 1951.

25. Van Dorn, Nicholas H., and DeYoung, John: A Comparison of Three Theoretical Methods of Calculating Span Load Distribution on Swept Wings. NACA TN 1476, 1947.

26. Kuessner, Hans Georg, and Billings, Heinz: Unsteady Flow. VI of Hydro- and Aerodynamics, Albert Betz, ed., ATI No. 72854, CADO, Wright-Patterson Air Force Base, May 1950, pp. 141-198.

27. Bickley, W. G., Comrie, L. J., et al.: Bessel Functions. Part II - Functions of Positive Integer Order. British Assoc. Mathematical Tables, vol. X, 1952.

28. Anon.: Table of the Struve Functions $L_\nu(x)$ and $H_\nu(x)$. Jour. Math. and Phys., vol. XXV, no. 3, Oct. 1946, pp. 252-259.

29. Karp, S. N., Shu, S. S., and Weil, H.: Aerodynamics of the Oscillating Airfoil in Compressible Flow. Tech. Rep. No. F-TR-1167-ND, Air Materiel Command, U. S. Air Force, Oct. 1947.

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